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# On the strong and very strong summability of orthogonal series

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**1.** Let $\{\varphi_n(x)\}$ be an orthonormal system on the interval (0, 1). We shall consider real orthogonal series

$$\sum_{n=0}^{\infty} c_n \varphi_n(x) \quad \text{with} \quad \sum_{n=0}^{\infty} c_n^2 < \infty. \tag{1.1}$$

By the Riesz—Fischer theorem series (1.1) converges in $L^2$ to a square-integrable function $f(x)$. Let us denote the partial sums of (1.1) by $s_n(x)$.

As introductory sample results we recall the following theorems:

Theorem A (A. ZYGMUND [15]). *If series* (1.1) $(C, 1)$*-summable almost everywhere then it is also strongly* $(C, 1)$*-summable almost everywhere, i.e.*

$$\frac{1}{n+1} \sum_{k=0}^{n} |s_k(x) - f(x)|^2 \to 0 \quad \textit{a.e.}$$

Theorem B (K. TANDORI [13]). *If*

$$\sum_{n=4}^{\infty} c_n^2 \log\log^2 n < \infty$$

*then series* (1.1) *is very strongly* $(C, 1)$*-summable on* (0, 1) *almost everywhere, i.e.*

$$\frac{1}{n+1} \sum_{k=0}^{n} |s_{\nu_k}(x) - f(x)|^2 \to 0$$

*for any increasing sequence* $\{\nu_k\}$ *of natural numbers on* (0, 1) *almost everywhere.*

Theorem C (K. TANDORI [12]). *There exist an orthonormal system* $\{\varphi_n(x)\}$ *and coefficients* $d_n$ *with* $\sum_{n=0}^{\infty} d_n^2 < \infty$ *such that the series*

$$\sum_{n=0}^{\infty} d_n \varphi_n(x)$$

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*is strongly* $(C, 1)$*-summable almost everywhere but it is nowhere very strongly* $(C, 1)$*-summable.*

In other words Theorem C states that the strong $(C, 1)$-summability does not imply the very strong $(C, 1)$-summability, generally.

The analogues of Theorems A and B for other summability methods have been proved individually. E.g. for Riesz-means J. MEDER [5] and L. LEINDLER [1], for $(C, \alpha>0)$-means G. SUNOUCHI [11], for Euler-means H. SCHWINN [9] and for generalized Abel-method L. LEINDLER [3] proved similar results.

On the other hand. F. MÓRICZ [6] proved that for an arbitrary regular Toeplitz T-summability method it is not true that if series (1.1) is T-summable then it is strongly T-summable, too.

The Móricz's result gives a reason for writing of a new paper, namely in the present paper we prove the analogues of Theorems A and B for a large class of general summability methods; and shall apply them to verify that the so-called generalized de la Vallée Poussin method also belongs to these summability methods. It will be easy to see that some of the above mentioned summability methods also belong to the class to be treated in Theorem 1. Roughly speaking one of the aims of the present paper is to verify that for a large class of summability methods the summability implies the strong summability for orthogonal series.

We mention that H. SCHWINN [10] also investigated the latter problem, and proved a slightly sharper result, but his proof quite differs from our one.

Theorem C shows that it cannot be expected that a general summability method should imply the very strong summability. But we shall show that if a coefficient-condition, e.g. of the form

$$\sum_{n=0}^{\infty} c_n^2 \varrho_n^2 < \infty \quad (\varrho_n \leqq \varrho_{n+1}),$$

implies the summability — as in Theorem B — then this condition will imply the very strong summability, too.

Let $\alpha := (\alpha_{nk})$ be a positive regular Toeplitz-matrix satisfying the usual conditions: $\alpha_{nk} \geqq 0$; $\lim\limits_{n\to\infty} \alpha_{nk} = 0$ $(k=0, 1, \ldots)$; $\lim\limits_{n\to\infty} \sum\limits_{k=0}^{\infty} \alpha_{nk} = 1$. We say that *series* (1.1) *is* $\alpha$*-summable to* $f(x)$ *if*

$$\alpha_n(x) := \sum_{k=0}^{\infty} \alpha_{nk} s_k(x) \to f(x)$$

*almost everywhere;* and it will be called *strongly* $\alpha$*-summable if*

$$\alpha_n |x| := \sum_{k=0}^{\infty} \alpha_{nk} |s_k(x) - f(x)|^2 \to 0$$

*almost everywhere; and very strongly $\alpha$-summable if for any increasing sequence $\{v_k\}$ of natural numbers*

$$\alpha_n\,|v;\,x| := \sum_{k=0}^{\infty} \alpha_{nk}\,|s_{v_k}(x)-f(x)|^2 \to 0$$

*holds almost everywhere.*

We say that an $\alpha$-summability method is an $N(\mu_m)$-*summability* if there exists an increasing sequence $\{\mu_m\}$ of natural numbers such that if series (1.1) is $\alpha$-summable then $s_{\mu_m}(x)\to f(x)$ always holds almost everywhere, i.e. the convergence of the partial sums $s_{\mu_m}(x)$ is a necessary condition of the $\alpha$-summability of series (1.1) for any orthonormal system $\{\varphi_n\}$ and for any coefficients $c_n$ with $\sum_{n=0}^{\infty} c_n^2<\infty$. It is known that the $(C, \alpha>0)$-methods and generalized Abel-methods (for the latter see L. Rempulska [7]) are $N(2^m)$-summability methods and the Euler-method is an $N(m^2)$-method (see O. A. Ziza [14] and H. Schwinn [8]).

Now we recall the definition of the generalized ordinary and strong de la Vallée Poussin summability methods (see [2]). Let $\lambda=\{\lambda_n\}$ be a nondecreasing sequence of natural numbers for which $\lambda_0=1$ and $\lambda_{n+1}-\lambda_n\leqq 1$. Series (1.1) is $(V, \lambda)$-*summable* if

$$V_n(\lambda;\,x) := \frac{1}{\lambda_n}\sum_{k=n-\lambda_n+1}^{n} s_k(x)\to f(x)$$

almost everywhere, *strongly* $(V, \lambda)$-*summable* if

$$V_n\,|\lambda;\,x| := \frac{1}{\lambda_n}\sum_{k=n-\lambda_n+1}^{n} |s_k(x)-f(x)|^2\to 0$$

almost everywhere; and *very strongly* $(V, \lambda)$-*summable* if for any increasing sequence $\{v_k\}$ of natural numbers

$$V_n\,|\lambda,\,v;\,x| := \frac{1}{\lambda_n}\sum_{k=n-\lambda_n+1}^{n} |s_{v_k}(x)-f(x)|^2\to 0$$

almost everywhere.

It is obvious that if $\lambda_n=n$ then the $(V, \lambda)$-means reduce to the $(C, 1)$-means, and if $\lambda_n=\left[\frac{n}{2}\right]$ $(n\geqq 2)$, where $[\beta]$ denotes the integer part of $\beta$, then we get the classical de la Vallée Poussin means.

In [2] we proved that for any $\lambda$ the $(V, \lambda)$-summability is an $N(v_m)$-summability with $v_0=1$ and $v_m:=\sum_{k=0}^{m-1}\lambda_{v_k}$, $m\geqq 1$; furthermore that if

$$\sum_{m=1}^{\infty}\Big\{\sum_{n=v_m+1}^{v_{m+1}} c_n^2\Big\}\log^2 m<\infty \tag{1.2}$$

then series (1.1) is $(V, \lambda)$-summable; moreover very strongly $(V, \lambda)$-summable.

**2.** Now we can formulate our theorems:

Theorem 1. *If a positive regular Toeplitz-matrix* $\alpha=(\alpha_{nk})$ *generates an* $N(\mu_m)$ *summability and satisfies the following additional conditions: there exist a natural number p and a positive M constant such that*

$$\alpha_{nk} \leqq M \sum_{i=-p}^{p} \alpha_{(\mu_{m-i})k} \quad \text{for} \quad \mu_{m-1} < n < \mu_m \tag{2.1}$$

*and*

$$\sum_{\nu=1}^{\infty} \sum_{i=\mu_{m-1}+1}^{\mu_m} \alpha_{\mu_\nu i} \leqq M \tag{2.2}$$

*hold for all m and k, then the* $\alpha$*-summability of series* (1.1) *implies its strong* $\alpha$*-summability.*

Theorem 2. *Under the assumptions of Theorem* 1, *if the following condition*

$$\sum_{n=1}^{\infty} c_n^2 \gamma_n^2 < \infty, \quad \gamma_n \leqq \gamma_{n+1}, \tag{2.3}$$

*implies the* $\alpha$*-summability of series* (1.1) *for any orthonormal system, then* (2.3) *also implies the very strong* $\alpha$*-summability of series* (1.1).

Using these theorems we verify the following theorems:

Theorem 3. *If series* (1.1) *is* $(V, \lambda)$*-summable then it is strongly* $(V, \lambda)$*-summable, too.*

Theorem D. *Condition* (1.2) *implies that series* (1.1) *is very strongly* $(V, \lambda)$ *summable.*

We remark that Theorem D was proved in [2] as we stated above, but its proof is totally different from to be given here.

**3.** We require the following lemma.

Lemma ([4], Lemma 3). *Let* $\varkappa>0$ *and* $\{\beta_n\}$ *be an arbitrary sequence of non-negative numbers. Assuming that the condition*

$$\sum_{n=1}^{\infty} \beta_n \Big\{\sum_{k=n}^{\infty} c_k^2\Big\}^{\varkappa} < \infty \tag{3.1}$$

*implies a "certain property* $T=T(\{s_n(x)\})$*" of the partial sums* $s_n(x)$ *of* (1.1) *for any orthonormal system, then* (3.1) *implies that the partial sums* $s_{\nu_n}(x)$ *of* (1.1) *also have the same property T for any increasing sequence* $\{\nu_n\}$, *i.e. if* (3.1)$\Rightarrow T(\{s_n(x)\})$ *then* (3.1)$\Rightarrow T(\{s_{\nu_n}(x)\})$ *for any increasing sequence* $\{\nu_n\}$.

**4.** Now we can prove our theorems. For the sake of brevity, from now on, convergence and summability have the meaning of convergence and summability almost everywhere in (0, 1).

Proof of Theorem 1. Since the $\alpha$-summability now implies the convergence of the partial sums $s_{\mu_m}(x)$, thus putting $\nu_k:=\mu_m$ for $k=\mu_{m-1}+1,\ \mu_{m-1}+2, \ldots$ $\ldots, \mu_m,\ m=1,2,\ldots;\ \nu_0=0$ and $\nu_1=1$; we can see by the following obvious inequality

$$\alpha_n |x| \leqq 2 \sum_{k=0}^{\infty} \alpha_{nk}\left(|s_k(x)-s_{\nu_k}(x)|^2+|s_{\nu_k}(x)-f(x)|^2\right)$$

and on account of the regularity of $\alpha$-summability, that in order to prove Theorem 1 it is enough to verify that

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} \alpha_{nk} |s_k(x)-s_{\nu_k}(x)|^2 = 0 \tag{4.1}$$

holds almost everywhere.

By (2.1) we have for any $\mu_{m-1}<n<\mu_m$

$$\sum_{k=0}^{\infty} \alpha_{nk} |s_k(x)-s_{\nu_k}(x)|^2 \leqq M \sum_{k=0}^{\infty} \sum_{i=-p}^{p} \alpha_{(\mu_{m-i})k} |s_k(x)-s_{\nu_k}(x)|^2, \tag{4.2}$$

therefore if we can prove

$$\lim_{m\to\infty} \sum_{k=0}^{\infty} \alpha_{\mu_m k} |s_k(x)-s_{\nu_k}(x)|^2 = 0 \tag{4.3}$$

almost everywhere, then, by (4.2), (4.1) will be proved.

An elementary calculation shows on account of (2.2) that

$$\sum_{m=1}^{\infty}\sum_{k=0}^{\infty} \alpha_{\mu_m k} \int_0^1 |s_k(x)-s_{\nu_k}(x)|^2\,dx \leqq \sum_{m=1}^{\infty}\sum_{i=1}^{\infty}\sum_{k=\mu_{i-1}+1}^{\mu_i} \alpha_{\mu_m k} \sum_{k=\mu_{i-1}+1}^{\mu_i} c_k^2 =$$

$$= \sum_{i=1}^{\infty}\sum_{k=\mu_{i-1}+1}^{\mu_i} c_k^2 \sum_{m=1}^{\infty}\sum_{k=\mu_{i-1}+1}^{\mu_i} \alpha_{\mu_m k} \leqq M \sum_{k=0}^{\infty} c_k^2 < \infty,$$

whence by B. Levi's theorem (4.3) follows.

This has completed the proof.

Proof of Theorem 2. Putting $\gamma_0=0$ then condition (2.3) and

$$\sum_{n=1}^{\infty} (\gamma_n^2-\gamma_{n-1}^2) \sum_{k=n}^{\infty} c_k^2 < \infty$$

are obviously equivalent. Hence we can already see that the statement of Theorem 2 is a consequence of Theorem 1 and our Lemma.

Proof of Theorem 3. In order to prove Theorem 3 it is enough to verify that if

$$\mu_0 = 0 \quad \text{and} \quad \mu_m := \sum_{k=0}^{m-1} \lambda_{\mu_k}, \tag{4.4}$$

then for this sequence $\{\mu_m\}$ conditions (2.1) and (2.2) are fulfilled.

Since $\lambda_{i+1}-\lambda_i \leqq 1$ for any $i$ and $\mu_m-\mu_{m-1}=\lambda_{\mu_{m-1}}$, therefore $\lambda_{\mu_m} \leqq 2\lambda_{\mu_{m-1}}$, whence for any $\mu_{m-1}<n<\mu_m$

$$\frac{1}{\lambda_n} \leqq \frac{1}{\lambda_{\mu_{m-1}}} \leqq \frac{2}{\lambda_{\mu_m}} \tag{4.5}$$

holds, which verifies that (2.1) also holds with $M=2$ and $p=1$. Since then for any $n$

$$\alpha_{nk} = \begin{cases} 0 & \text{for} \quad k < n+1-\lambda_n \quad \text{and} \quad k > n, \\ \dfrac{1}{\lambda_n} & \text{for} \quad n+1-\lambda_n \leqq k \leqq n; \end{cases} \tag{4.6}$$

and thus by (4.5) for $\mu_{m-1}<n<\mu_m$

$$\alpha_{nk} \leqq 2(\alpha_{\mu_{m-1}k}+\alpha_{\mu_m k})$$

always holds because $\mu_{m-1}+1-\lambda_{\mu_{m-1}} \leqq n+1-\lambda_n$ by $\lambda_{i+1}-\lambda_i \leqq 1$.

Namely if $n+1-\lambda_n \leqq k \leqq \mu_{m-1}$ then

$$\alpha_{nk} = \frac{1}{\lambda_n} \leqq \frac{1}{\lambda_{\mu_{m-1}}} = \alpha_{\mu_{m-1}k}$$

and if $\mu_{m-1}<k \leqq n<\mu_m$ then by (4.5)

$$\alpha_{nk} = \frac{1}{\lambda_n} \leqq \frac{2}{\lambda_{\mu_m}} = 2\alpha_{\mu_m k}$$

hold.

Next we show that (2.2) is also fulfilled for the $\alpha$-matrix given by (4.6) and for sequence $\{\mu_m\}$ defined under (4.4).

By (4.6) it is clear that if $\nu \leqq m-1$ then

$$\sum_{i=\mu_{m-1}+1}^{\mu_m} \alpha_{\mu_\nu i} = 0 \quad (\mu_\nu < i), \tag{4.7}$$

and if $\nu \geqq m$ then

$$\sum_{i=\mu_{m-1}+1}^{\mu_m} \alpha_{\mu_\nu i} \leqq \frac{1}{\lambda_{\mu_\nu}}[\mu_m-(\mu_\nu-\lambda_{\mu_\nu})]^+ =: A_{m,\nu}, \tag{4.8}$$

where $[\beta]^+$ denotes the positive part of $\beta$. On account of the definition $\{\mu_m\}$ and the property $\lambda_{\mu_m} \leqq 2\lambda_{\mu_{m-1}}$ we can verify that for any $\nu \geqq m$

$$A_{m,\nu} \leqq \left(\frac{1}{2}\right)^{\nu-m} \tag{4.9}$$

holds. Namely an easy calculation shows that

$$A_{m,\nu} = \left[1-\frac{\mu_\nu-\mu_m}{\lambda_{\mu_\nu}}\right]^+ = [1-\lambda_{\mu_\nu}^{-1}\sum_{k=m}^{\nu-1}\lambda_{\mu_k}]^+ \leqq$$

$$\leqq [1-(2\lambda_{\mu_{\nu-1}})^{-1}\sum_{k=m}^{\nu-1}\lambda_{\mu_k}]^+ = \left[\frac{1}{2}-(2\lambda_{\mu_{\nu-1}})^{-1}\sum_{k=m}^{\nu-2}\lambda_{\mu_k}\right]^+ \leqq$$

$$\leqq \left[\frac{1}{4}-(4\lambda_{\mu_{\nu-2}})^{-1}\sum_{k=m}^{\nu-3}\lambda_{\mu_k}\right]^+ \leqq \ldots \leqq \left(\frac{1}{2}\right)^{\nu-m}.$$

Collecting the results of (4.7), (4.8) and (4.9) we get (2.2) with $M=2$. So we can apply Theorem 1 which obviously proves Theorem 3.

Proof of Theorem D. Let

$$\beta_0 = \beta_1 = \ldots = \beta_{\mu_1} = 1;$$

$$\beta_n := \frac{\log m}{(\mu_{m+1}-\mu_m)m} \quad \text{for} \quad \mu_m < n \leqq \mu_{m+1}, \quad m = 1, 2, \ldots;$$

and $\varkappa=2$. A standard calculation shows that for these $\beta_n$ and $\varkappa$ (3.1) holds if and only if (1.2) is fulfilled. Namely if (3.1) holds then by

$$\sum_{n=1}^{\infty}\beta_n\sum_{k=n}^{\infty}c_k^2 \geqq \sum_{m=1}^{\infty}\sum_{n=\mu_m+1}^{\mu_{m+1}}\beta_n\sum_{k=\mu_{m+1}+1}^{\infty}c_k^2 =$$

$$= \sum_{m=1}^{\infty}\frac{\log m}{m}\sum_{\nu=m+1}^{\infty}\sum_{k=\mu_\nu+1}^{\mu_{\nu+1}}c_k^2 = \sum_{\nu=2}^{\infty}\sum_{k=\mu_\nu+1}^{\mu_{\nu+1}}c_k^2\sum_{m=1}^{\nu-1}\frac{\log m}{m}$$

(1.2) also holds. Conversely if (1.2) is fulfilled then the following inequalities

$$\sum_{m=1}^{\infty}\Big(\sum_{k=\mu_m+1}^{\mu_{m+1}}c_k^2\Big)\log^2 m \geqq \frac{1}{4}\sum_{m=3}^{\infty}\Big(\sum_{k=\mu_m+1}^{\mu_{m+1}}c_k^2\Big)\sum_{\nu=2}^{m-1}\frac{\log\nu}{\nu} =$$

$$= \frac{1}{4}\sum_{\nu=2}^{\infty}\frac{\log\nu}{\nu}\sum_{m=\nu+1}^{\infty}\sum_{k=\mu_m+1}^{\mu_{m+1}}c_k^2 \geqq \frac{1}{4}\sum_{\nu=2}^{\infty}\frac{\log(\nu+1)}{\nu+1}\sum_{k=\mu_{\nu+1}+1}^{\infty}c_k^2 =$$

$$= \frac{1}{4}\sum_{m=3}^{\infty}\frac{\log m}{m}\sum_{k=\mu_m+1}^{\infty}c_k^2 \geqq \frac{1}{4}\sum_{m=3}^{\infty}\sum_{n=\mu_m+1}^{\mu_{m+1}}\beta_n\sum_{k=n}^{\infty}c_k^2 = \frac{1}{4}\sum_{n=\mu_3+1}^{\infty}\beta_n\sum_{k=n}^{\infty}c_k^2$$

prove (3.1).

On account of this equivalence and our Lemma the statement of Theorem D is proved.

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